# Optical Soliton, and Propagating Delta Function

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## Abstract

A detailed derivation of Optical Solitons, and experimental data suggest that an Optical Soliton may be modeled by a propagating Delta Function.

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**Contents**

# *Optical Solitons*

Optical Solitons are pulses enveloping an optical frequency carrier that keep their shape as they propagate through the optical fibers.

A wave packet fed into an optical fiber, evolves into a soliton if its group velocity dispersion is balanced by its self-phase modulation in the glass.

That balance is attained when the frequency $\omega$ of the carrier under the pulse, remains the same before the peak of the pulse, and beyond the peak of the pulse.

The frequency $\omega$, is the time derivative of the Non-linear phase function of the pulse in the optical fiber.

the phase function is necessary to obtain an expression for the wave packet that is used to derive the Non-Linear Schrödinger Equation, the Soliton's equation

The phase function depends on the frequency $\omega$, and on the refractive index $n(\omega, E_0)$, where $E_0$ is the Amplitude of the wave packet, and may be approximated by a Taylor polynomial in the variables $\omega$, and $E_0$.

That approximation is nowhere to be found in the literature. authors seem to know about Taylor polynomials in one variable, but not in two variables.

Consequently, the literature has no substantiated derivation of the Non-Linear Schrödinger Equation.

We aim to show that Solitons may be represented by a Delta function, but this requires that generation of Solitons be clearly understood.

In section 1, we obtain the Taylor approximation for the nonlinear phase function, and the expression for the wave packet

In section 2, we derive the Non-Linear Schrödinger equation.

In section 3, we discuss dispersion and phase modulation of a Gaussian pulse, in order to gain insight about pulses in general, and solitons in particular.

In section 4, we obtain the Non-linear Schrödinger equation modified for solitons, and in section 5 the equation for the Soliton's envelope equation.

in section 6, we present the known fundamental Soliton solution, in a form that will enable us to establish that it may be represented by a Delta function.

We do that in the second part of the paper, in sections 7-13.

# 1.

# Non-Linear Phase Function

Consider a wave packet of light frequencies propagating in an optical fiber,

$$\psi(z,t) = \int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i(\omega t-\beta z)}d\omega .$$

The wave packet may be written as

$$\psi(z,t) = e^{-i\omega_0 t}\int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i(\omega_0 t-\beta z)}e^{i\omega t}d\omega$$

Then, it is an envelope $\int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i(\omega_0 t-\beta z)}e^{i\omega t}d\omega$ modulating a carrier with frequency $\omega_0$.

The phase function is

$$\omega_0 t - \beta z .$$

In the air, $\beta$ is a constant $\beta_0$. In the optical fiber, $\beta$ is a nonlinear function of $\omega$, and of the light intensity $I$. We aim to express $\beta$ in terms of $\omega$, and $I$.

Let $\vec{E}$, and $\vec{B}$ be the electromagnetic fields of the wave packet.

By Faraday's Law, $\nabla \times \vec{E} = -\partial_t \vec{B}$,

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times (-\partial_t \vec{B}),$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\partial_t \nabla \times \vec{B},$$

$$= -\partial_t \nabla \times (\vec{M} + \mu_0 \vec{H})$$

Assuming a scalar dielectric coefficient $\varepsilon = \varepsilon_0 \varepsilon_r$, and $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon}$.

Assuming no electric charges, $\rho = 0$, and $\nabla \cdot \vec{E} = 0$.

Assuming no magnetic charges, $\vec{M} = 0$, and

$$-\nabla^2 \vec{E} = -\mu_0 \partial_t \nabla \times \vec{H}$$

By Ampere's Law, $\nabla \times \vec{H} = \vec{J} + \partial_t \vec{D}$.

Assuming no current, $\vec{J} = 0$, $\nabla \times \vec{H} = \partial_t \vec{D}$, and

$$\nabla^2 \vec{E} = \mu_0 \partial_t^2 \vec{D}$$

$$= \mu_0 \partial_t^2 (\varepsilon_0 \varepsilon_r \vec{E})$$

$$= \varepsilon_0 \mu_0 \partial_t^2 (\varepsilon_r \vec{E})$$

Assuming a Plane Electromagnetic wave propagating in the $z$ direction, $\vec{E} = \left[ E(z,t), \quad 0, \quad 0 \right]$, and

$$\partial_z^2 E = \varepsilon_0 \mu_0 \partial_t^2 (\varepsilon_r E).$$

The refractive index is

$$n = \sqrt{\varepsilon_r} \, ,$$

and

$$\beta = \frac{\omega_0}{c_0} n \, ,$$

where

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

is the speed of the light in the vacuum.

Since

$$\varepsilon_r = n^2,$$

and since

$$\varepsilon_r = 1 + \chi,$$

where $\chi$ is the susceptibility of the optical fiber, we have

$$n^2 = 1 + \chi.$$

In a Non-Linear Optical Medium, $\chi E$ may depend on powers of the electric field $E$

$$\chi E = \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + ... + \chi^{(N)} E^N,$$

where the coefficients $\chi^{(1)}, \chi^{(2)} ... \chi^{(N)}$ are very small. In Optical Fibers, we put $\chi^{(2)} = 0$, and assume a cubic nonlinearity

$$\chi E = \chi^{(1)} E + \chi^{(3)} E^3.$$

Therefore,

$$n^2 E = (1 + \chi) E$$

$$= (1 + \chi^{(1)}) E + \chi^{(3)} E^3.$$

Substituting a harmonic Electric field

$$E = E_0(z,t) \cos(\omega t - kz),$$

$$n^2 E_0 \cos(\omega t - kz) = (1 + \chi^{(1)}) E_0 \cos(\omega t - kz) + \chi^{(3)} E_0^3 \cos^3(\omega t - kz)$$

Since $E_0 \neq 0$,

$$n^2\cos(\omega t - kz) = (1+\chi^{(1)})\cos(\omega t - kz) + \chi^{(3)}E_0^2\cos^3(\omega t - kz)$$

Now,

$$\cos^3(\omega t - kz) = \frac{1}{2^3}(e^{i(wt-kz)} + e^{-i(wt-kz)})^3$$

$$= \frac{1}{2^3}(e^{3i(wt-kz)} + 3e^{i(wt-kz)} + 3e^{-i(wt-kz)} + e^{-3i(wt-kz)})$$

In Optical Fibers, the third harmonics $3(wt-kz)$ is negligible, and

$$\cos^3(\omega t - kz) \approx \frac{3}{2^3}(e^{i(wt-kz)} + e^{-i(wt-kz)})$$

$$= \frac{3}{4}\cos(\omega t - kz)$$

Substituting into the equation for $n^2$,

$$n^2\cos(\omega t - kz) = (1+\chi^{(1)})\cos(\omega t - kz) + \frac{3}{4}\chi^{(3)}E_0^2\cos(\omega t - kz)$$

Since $\cos(\omega t - kz)$ does not vanish identically,

$$n^2 = (1+\chi^{(1)}) + \frac{3}{4}\chi^{(3)}E_0^2.$$

Denoting

$$\sqrt{1+\chi^{(1)}} \equiv n_0,$$

we have

$$\frac{n}{n_0} = \left(1 + \frac{3}{4}\frac{\chi^{(3)}}{n_0^2}E_0^2\right)^{\frac{1}{2}}.$$

Since $\chi^{(3)}$ is very small,

$$\approx 1 + \frac{3}{8}\frac{\chi^{(3)}}{n_0^2}E_0^2 .$$

Thus,

$$n \approx n_0 + \frac{3}{8}\frac{\chi^{(3)}}{n_0}E_0^2 .$$

The dependence of $n$ on the frequency $\omega$ is implicit, and we shall expand

$$n = n(\omega, E_0)$$

in a Taylor Polynomial of the second order about the point

$$\omega = \omega_0, \qquad E_0 = 0 .$$

$$n(\omega, E_0) = n(\omega_0, 0) + \left.\frac{\partial n}{\partial \omega}\right|_{\substack{\omega=\omega_0\\ E_0=0}}(\omega - \omega_0) + \left.\frac{\partial n}{\partial E_0}\right|_{\substack{\omega=\omega_0\\ E_0=0}} E_0 +$$

$$+\frac{1}{2}\left.\frac{\partial^2 n}{\partial \omega^2}\right|_{\substack{\omega=\omega_0\\ E_0=0}}(\omega - \omega_0)^2 + \left.\frac{\partial^2 n}{\partial \omega \partial E_0}\right|_{\substack{\omega=\omega_0\\ E_0=0}}(\omega - \omega_0)E_0 + \frac{1}{2}\left.\frac{\partial^2 n}{\partial E_0^2}\right|_{\substack{\omega=\omega_0\\ E_0=0}} E_0^2$$

Here,

$$\left.\frac{\partial n}{\partial E_0}\right|_{\substack{\omega=\omega_0\\ E_0=0}} = \left.\frac{3}{4}\frac{\chi^{(3)}}{n_0}E_0\right|_{\substack{\omega=\omega_0\\ E_0=0}} = 0,$$

$$\left.\frac{\partial^2 n}{\partial \omega \partial E_0}\right|_{\substack{\omega=\omega_0\\ E_0=0}} = \left.\frac{3}{4}\left(\partial_\omega \frac{\chi^{(3)}}{n_0}\right)E_0\right|_{\substack{\omega=\omega_0\\ E_0=0}} = 0$$

$$\frac{1}{2}\left.\frac{\partial^2 n}{\partial E_0^2}\right|_{\substack{\omega=\omega_0\\ E_0=0}} = \frac{3}{8}\frac{\chi^{(3)}}{n_0}$$

Therefore,

$$n(\omega, E_0) = n(\omega_0, 0) + \frac{\partial n}{\partial \omega}\bigg|_{\substack{\omega=\omega_0 \\ E_0=0}} (\omega - \omega_0) + \frac{1}{2}\frac{\partial^2 n}{\partial \omega^2}\bigg|_{\substack{\omega=\omega_0 \\ E_0=0}} (\omega - \omega_0)^2 + \frac{3}{8}\frac{\chi^{(3)}}{n_0} E_0^2$$

Multiplying by $k_0 = \dfrac{\omega_0}{c_0}$,

$$\underbrace{k_0 n(\omega, E_0)}_{\beta(\omega, E_0)} = \underbrace{k_0 n(\omega_0, 0)}_{\beta_0} + \underbrace{\frac{\partial (k_0 n)}{\partial \omega}\bigg|_{\substack{\omega=\omega_0 \\ E_0=0}}}_{\frac{\partial \beta}{\partial \omega} \equiv \beta_1} (\omega - \omega_0) + \frac{1}{2}\underbrace{\frac{\partial^2 (k_0 n)}{\partial \omega^2}\bigg|_{\substack{\omega=\omega_0 \\ E_0=0}}}_{\frac{\partial^2 \beta}{\partial \omega^2} \equiv \beta_2} (\omega - \omega_0)^2 + \frac{\omega_0}{c_0}\frac{3}{8}\frac{\chi^{(3)}}{n_0} E_0^2$$

Denoting

$$\frac{3}{4}\frac{\chi^{(3)}}{c_0 \varepsilon_0 n_0^2} \equiv n_2,$$

we have,

$$\beta(\omega, E_0) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 E_0^2.$$

Denoting

$$I \equiv \frac{1}{2} c_0 \varepsilon_0 n_0 E_0^2,$$

the refractive index is

$$n(\omega, I) \approx n_0 + n_2 I,$$

and

$$\beta(\omega, I) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \frac{\omega_0}{c_0} n_2 I.$$

Therefore, the phase function is

$$\omega_0 t - \left[\beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \frac{\omega_0}{c_0} n_2 I\right] z,$$

and the wave packet is

$$\psi(z,t) = e^{-i\omega_0 t} \int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left(\omega_0 t - [\beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{\omega_0}{c_0} n_2 I]z\right)} e^{i\omega t} d\omega \,.$$

This wave packet leads to the Non-Linear Schrödinger equation.

# 2.

# Non-Linear Schrodinger Equation

The wave packet

$$\psi(z,t) = e^{-i\omega_0 t}\int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i\left(\omega_0 t-[\beta_0+\beta_1(\omega-\omega_0)+\frac{1}{2}\beta_2(\omega-\omega_0)^2+\frac{\omega_0}{c_0}n_2 I]z\right)}e^{i\omega t}d\omega$$

$$= e^{i(\omega_0 t-\beta_0 z)}\underbrace{\int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)}d\omega}_{E_0(z,t)}$$

is an envelope

$$E_0(z,t) = \int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)}d\omega$$

modulating the carrier $e^{i(\omega_0 t-\beta_0 z)}$.

We show that the envelope satisfies the Non-Linear Schrodinger Equation.

$$\boxed{\partial_z E_0 + \beta_1\partial_t E_0 - i\frac{1}{2}\beta_2\partial_{tt}E_0 + i\frac{1}{2}\omega_0\varepsilon_0 n_0 n_2\left|E_0\right|^2 E_0 = 0}$$

$$\partial_z E_0(z,t) = \partial_z\int_{\omega=-\infty}^{\omega=\infty} A(\omega)e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)}d\omega$$

$$= -i \int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)} \left[\beta_1[\omega-\omega_0]+\frac{1}{2}\beta_2[\omega-\omega_0]^2+\frac{\omega_0}{c_0}n_2 I\right] d\omega$$

$$= -\beta_1 \underbrace{i \int_{\omega=-\infty}^{\omega=\infty} (\omega-\omega_0) A(\omega) e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)} d\omega}_{\partial_t E_0}$$

$$+i\frac{1}{2}\beta_2 \underbrace{i^2 \int_{\omega=-\infty}^{\omega=\infty} (\omega-\omega_0)^2 A(\omega) e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)} d\omega}_{\partial_{tt} E_0}$$

$$-i \underbrace{\frac{\omega_0}{c_0} n_2 I}_{\frac{1}{2}\omega_0 n_0 n_2 |E_0|^2} \underbrace{\int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left((t-\beta_1 z)(\omega-\omega_0)-\frac{1}{2}\beta_2(\omega-\omega_0)^2 z-\frac{\omega_0}{c_0}n_2 Iz\right)} d\omega}_{E_0}$$

Therefore,

$$\partial_z E_0 = -\beta_1 \partial_t E_0 + i\frac{1}{2}\partial_{tt} E_0 - i\frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 \left|E_0\right|^2 E_0 . \square$$

We proceed to explore the possibility of a Soliton solution for the Non-Linear Schrödinger equation.

To that end, we follow the propagation of a Gaussian pulse through the optical fiber.

# 3.

# Dispersion, and Phase Modulation of a Gaussian Pulse

Since the wave packet is

$$\psi(z,t) = e^{-i\omega_0 t}\int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left(\omega_0 t - [\beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{\omega_0}{c_0} n_2 I]z\right)} e^{i\omega t} d\omega\,,$$

we have,

$$\psi(0,t) = \int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\omega t} d\omega\,.$$

Therefore,

$$A(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{t=\infty} \psi(0,t) e^{-i\omega t} dt\,.$$

Let $\psi(0,t)$ be a Gaussian pulse enveloping the carrier $e^{i\omega_0 t}$,

$$\psi(0,t) = Ce^{-\frac{t^2}{2\sigma_t^2}} e^{i\omega_0 t}.$$

Then,

$$A(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{t=\infty} Ce^{-\frac{t^2}{2\sigma_t^2}} e^{i\omega_0 t} e^{-i\omega t} dt$$

$$= C\frac{1}{2\pi}\int_{t=-\infty}^{t=\infty} e^{-\frac{1}{2}\left[\frac{t^2}{\sigma_t^2}-2i(\omega_0-\omega)t\right]}dt$$

$$= C\frac{1}{2\pi}\int_{t=-\infty}^{t=\infty} e^{-\frac{1}{2}\left[\frac{t}{\sigma_t}-i\sigma_t(\omega_0-\omega)\right]^2-\frac{1}{2}\sigma_t^2(\omega_0-\omega)^2}dt$$

$$= C\frac{1}{2\pi}\left(\int_{t=-\infty}^{t=\infty} e^{-\frac{1}{2}\left[\frac{t}{\sigma_t}-i\sigma_t(\omega_0-\omega)\right]^2}dt\right)e^{-\frac{1}{2}\sigma_t^2(\omega_0-\omega)^2}$$

Put

$$\tau = \frac{t}{\sigma_t} - i\frac{1}{2}\sigma_t(\omega_0 - \omega)$$

$$d\tau = \frac{1}{\sigma_t}dt$$

Then,

$$A(\omega) = C\frac{1}{2\pi}\sigma_t \underbrace{\int_{\tau=-\infty}^{\tau=\infty} e^{-\frac{1}{2}\tau^2}d\tau}_{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sigma_t^2(\omega_0-\omega)^2}$$

$$= C\frac{1}{\sqrt{2\pi}}\sigma_t e^{-\frac{1}{2}\sigma_t^2(\omega-\omega_0)^2}.$$

Denoting

$$\omega - \omega_0 \equiv \tilde{\omega},$$

$$A(\omega) = C\frac{1}{\sqrt{2\pi}}\sigma_t e^{-\frac{1}{2}\sigma_t^2\tilde{\omega}^2}$$

The wave packet is

$$\psi(z,t) = e^{-i\omega_0 t} \int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left(\omega_0 t - [\beta_0 + \beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \frac{\omega_0}{c_0} n_2 I]z\right)} e^{i\omega t} d\omega$$

$$= e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)} \int_{\omega=-\infty}^{\omega=\infty} A(\omega) e^{i\left((t-\beta_1 z)(\omega-\omega_0) - \frac{1}{2}\beta_2(\omega-\omega_0)^2 z\right)} d\omega$$

$$= e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)} \int_{\tilde{\omega}=-\infty}^{\tilde{\omega}=\infty} A(\omega) e^{i\left((t-\beta_1 z)\tilde{\omega} - \frac{1}{2}\beta_2 \tilde{\omega}^2 z\right)} d\tilde{\omega}$$

Plugging in the $A(\omega)$ of a Gaussian Pulse,

$$\psi(z,t) = C \frac{1}{\sqrt{2\pi}} \sigma_t e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)} \int_{\tilde{\omega}=-\infty}^{\tilde{\omega}=\infty} e^{-\frac{1}{2}\sigma_t^2 \tilde{\omega}^2} e^{i\left((t-\beta_1 z)\tilde{\omega} - \frac{1}{2}\beta_2 \tilde{\omega}^2 z\right)} d\tilde{\omega}$$

$$= C \frac{1}{\sqrt{2\pi}} \sigma_t e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)} \int_{\tilde{\omega}=-\infty}^{\tilde{\omega}=\infty} e^{-\frac{1}{2}\tilde{\omega}^2\left(\sigma_t^2 + i\beta_2 z\right) + i\left((t-\beta_1 z)\tilde{\omega}\right)} d\tilde{\omega}.$$

The exponent of the integrand is

$$-\frac{1}{2}\tilde{\omega}^2\left(\sigma_t^2 + i\beta_2 z\right) + i(t-\beta_1 z)\tilde{\omega} =$$

$$= -\frac{1}{2}\underbrace{\left(\tilde{\omega}\left(\sigma_t^2 + i\beta_2 z\right)^{\frac{1}{2}} - i\left(\sigma_t^2 + i\beta_2 z\right)^{-\frac{1}{2}}(t-\beta_1 z)\right)}_{\hat{\omega}}^2 - \frac{1}{2}\frac{(t-\beta_1 z)^2}{\sigma_t^2 + i\beta_2 z}$$

Denote

$$\hat{\omega} = \tilde{\omega}\left(\sigma_t^2 + i\beta_2 z\right)^{\frac{1}{2}} - i\left(\sigma_t^2 + i\beta_2 z\right)^{-\frac{1}{2}}(t-\beta_1 z)$$

Then,

$$d\hat{\omega} = \left(\sigma_t^2 + i\beta_2 z\right)^{\frac{1}{2}} d\tilde{\omega},$$

and the wave packet is

$$\psi(z,t) = \frac{C}{\sqrt{2\pi}}\sigma_t e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)}\left(\sigma_t^2 + i\beta_2 z\right)^{-\frac{1}{2}} \underbrace{\int_{\hat{\omega}=-\infty}^{\hat{\omega}=\infty} e^{-\frac{1}{2}\hat{\omega}^2} d\hat{\omega}}_{\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(t-\beta_1 z)^2}{\sigma_t^2 + i\beta_2 z}}$$

$$= C\sigma_t e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)}\left(\sigma_t^2 + i\beta_2 z\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{(t-\beta_1 z)^2}{\sigma_t^2 + i\beta_2 z}}$$

Simplifying,

$$\frac{1}{\left(\sigma_t^2 + i\beta_2 z\right)^{\frac{1}{2}}} = \frac{\left(\sigma_t^2 - i\beta_2 z\right)^{\frac{1}{2}}}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{2}}}\left(\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{2}} e^{-i\arctan\left(\frac{\beta_2 z}{\sigma_t^2}\right)}\right)^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{1}{2}i\arctan\left(\frac{\beta_2 z}{\sigma_t^2}\right)}}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{4}}}$$

$$e^{-\frac{1}{2}\frac{(t-\beta_1 z)^2}{\sigma_t^2 + i\beta_2 z}} = e^{-\frac{1}{2}\frac{\sigma_t^2 - i\beta_2 z}{\sigma_t^4 + \beta_2^2 z^2}(t-\beta_1 z)^2}$$

$$= e^{\frac{-1}{2}\frac{\sigma_t^2}{\sigma_t^4 + \beta_2^2 z^2}(t-\beta_1 z)^2}\, e^{\frac{1}{2}i\frac{\beta_2 z}{\sigma_t^4 + \beta_2^2 z^2}(t-\beta_1 z)^2}$$

Hence,

$$\psi(z,t) = C\sigma_t e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z\right)} \frac{e^{-\frac{1}{2} i \arctan\left(\frac{\beta_2 z}{\sigma_t^2}\right)}}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{4}}} e^{\frac{1}{2}\frac{-\sigma_t^2}{\sigma_t^4+\beta_2^2 z^2}(t-\beta_1 z)^2} e^{\frac{1}{2} i \frac{\beta_2 z}{\sigma_t^4+\beta_2^2 z^2}(t-\beta_1 z)^2}$$

$$= \frac{C\sigma_t e^{\frac{-\sigma_t^2}{\sigma_t^4+\beta_2^2 z^2}(t-\beta_1 z)^2}}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{4}}} e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z - \frac{1}{2}\arctan\left(\frac{\beta_2 z}{\sigma_t^2}\right) + \frac{1}{2}\frac{\beta_2 z}{\sigma_t^4+\beta_2^2 z^2}(t-\beta_1 z)^2\right)}$$

The Power of the Pulse is proportional to

$$\left|\psi(z,t)\right|^2 = \frac{C^2 \sigma_t^2 e^{\frac{-2\sigma_t^2}{\sigma_t^4+\beta_2^2 z^2}(t-\beta_1 z)^2}}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{2}}}.$$

Amplification along optical fibers, renders them loss-less medium, and the pulse power does not dissipate with $z$. Alternatively, we may take

$$1 \gg \frac{\beta_2^2 z^2}{\sigma_t^4}.$$

Then,

$$\psi(z,t) \approx C e^{\frac{-1}{\sigma_t^2}(t-\beta_1 z)^2} e^{i\left(\omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z - \frac{1}{2}\arctan\left(\frac{\beta_2 z}{\sigma_t^2}\right) + \frac{\beta_2 z}{\sigma_t^4}(t-\beta_1 z)^2\right)}.$$

Hence,

$$\left|\psi(z,t)\right|^2 \approx C^2 e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2},$$

and the Gaussian Pulse power is

$$P(t) = P_0 e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2}.$$

The intensity of the field in an optical fiber is

$$I = \frac{P(t)}{A_{eff}}$$

$$= \frac{P_0 e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2}}{A_{eff}}$$

where $A_{eff}$ is the effective area of the fiber's cross section.

Therefore, the phase of $\psi(z,t)$ is

$$\varphi(t) = \omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 I z - \frac{1}{2}\arctan\frac{\beta_2 z}{\sigma_t^2} + \frac{\beta_2 z}{\sigma_t^4}(t - \beta_1 z)^2 =$$

$$= \omega_0 t - \beta_0 z - \frac{\omega_0}{c_0} n_2 \frac{P_0 e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2}}{A_{eff}} z - \frac{1}{2}\arctan\frac{\beta_2 z}{\sigma_t^2} + \frac{\beta_2 z}{\sigma_t^4}(t - \beta_1 z)^2.$$

The frequency of the carrier is

$$\omega(t) = \partial_t \varphi$$

$$= \omega_0 - \frac{\omega_0}{c_0} n_2 \frac{P_0}{A_{eff}} z \left( -\frac{4}{\sigma_t^2}(t - \beta_1 z) \right) e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2} + \frac{\beta_2 z}{\sigma_t^4}(t - \beta_1 z)$$

$$= \omega_0 + \left( 4\frac{\omega_0}{c_0} n_2 \frac{P_0}{A_{eff}} z e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2} + \frac{\beta_2 z}{\sigma_t^2} \right) \frac{(t - \beta_1 z)}{\sigma_t^2}$$

The frequency of the carrier is $\omega_0$, only at the peak of the pulse

That is, at

$$t = \beta_1 z.$$

Otherwise, the frequency depends on the sign of

$$4\frac{\omega_0}{c_0} n_2 \frac{P_0}{A_{eff}} z e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2} + \frac{\beta_2 z}{\sigma_t^2}.$$

The two terms in the sum have opposite signs, because in Optical Fibers

$$n_2 > 0,$$

and

$$\beta_2 < 0.$$

There are three cases

(I) <u>Group Velocity Dispersion</u>

If

$$4\frac{\omega_0}{c_0} n_2 \frac{P_0}{A_{eff}} z e^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2} + \frac{\beta_2 z}{\sigma_t^2} > 0.$$

Then,

The frequency decreases before the peak of the pulse, and the carrier vibrations get dispersed

The frequency increases beyond the peak of the pulse, and the carrier vibrations get compressed.

(II) <u>Self-Phase Modulation</u>

If

$$4\frac{\omega_0}{c_0}n_2\frac{P_0}{A_{eff}}ze^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2}+\frac{\beta_2 z}{\sigma_t^2}<0.$$

Then,

The frequency increases before the peak of the pulse, and the carrier vibrations get compressed

The frequency decreases beyond the peak of the pulse, and the carrier vibrations get dispersed.

(III) Optical Soliton

If

$$4\frac{\omega_0}{c_0}n_2\frac{P_0}{A_{eff}}ze^{\frac{-2}{\sigma_t^2}(t-\beta_1 z)^2}+\frac{\beta_2 z}{\sigma_t^2}=0.$$

Then,

The frequency remains the same $\omega_0$ before the peak of the pulse, and beyond the peak of the pulse.

the carrier vibrations dispersion is balanced by their compression.

We will modify the Non-Linear Schrödinger equation to generate a Soliton solution.

# 4.

# Non-Linear Schrodinger Equation for Solitons

The condition for the evolution of a Gaussian pulse into an Optical Soliton, involves a transformed time

$$\frac{t-\beta_1 z}{\sigma_t}$$

where $\sigma_t$ is the standard deviation of the Gaussian distribution.

Thus, we should expect an optical Soliton time to be of the form

$$\frac{t-\beta_1 z}{T},$$

where $T$ is the time-width of the Soliton.

But Soliton width which is not known at this point, can be introduced in a natural way after further developments.

Rather than introduce an unknown $T$ ,we transform the time $t$ to

$$\tau = t - \beta_1 z\,.$$

Then, the Soliton's envelope moves at the group velocity

$$v_g = \frac{1}{\beta_1},$$

The Non-Linear Schrödinger equation

$$\partial_z E_0 + \beta_1 \partial_t E_0 - i\frac{1}{2}\beta_2 \partial_{tt} E_0 + i\frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 \left|E_0\right|^2 E_0 = 0,$$

will transform to an equation that depends on $T$.

The modulating envelope is

$$E_0(z,t) = f(z,\tau)$$

Then,

$$\frac{\partial E_0}{\partial z} = \frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial z}}_{1} + \frac{\partial f}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial z}}_{-\beta_1}$$

$$= \partial_z f - \beta_1 \partial_\tau f$$

$$\frac{\partial E_0}{\partial t} = \frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{0} + \frac{\partial f}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial t}}_{1}$$

$$= \partial_\tau f$$

$$\frac{\partial^2 E_0}{\partial t^2} = \frac{\partial(\partial_\tau f)}{\partial z}\underbrace{\frac{\partial z}{\partial t}}_{0} + \frac{\partial(\partial_\tau f)}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial t}}_{1}$$

$$= \partial_{\tau\tau} f$$

Hence, the equation transforms to

$$\underbrace{\partial_z E_0}_{\partial_z f - \beta_1 \partial_\tau f} + \beta_1 \underbrace{\partial_t E_0}_{\partial_\tau f} - i\frac{1}{2}\beta_2 \underbrace{\partial_{tt} E_0}_{\partial_{\tau\tau} f} + i\frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 \underbrace{\left|E_0\right|^2 E_0}_{\left|f\right|^2 f} = 0$$

That is,

$$\partial_z f - i\frac{1}{2}\beta_2 \partial_{\tau\tau} f + i\frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 \left|f\right|^2 f = 0.$$

# 5.

# The Soliton's Envelope Equation

The Non-Linear Schrödinger equation

$$\partial_z f - i\frac{1}{2}\beta_2 \partial_{\tau\tau} f + i\frac{1}{2}\omega_0 \varepsilon_0 n_0 n_2 \left|f\right|^2 f = 0.$$

describes the propagation of a wave packet envelope in the optical fiber, in a coordinate system $(z, \tau)$ that moves with the pulse peak.

Since a Soliton's envelope does not depend on $z$, a Soliton solution has the form

$$f(z, \tau, a) = au(\tau)e^{i\phi(z)},$$

where

$u(\tau)$ is the Soliton's envelope, $\tau = t - \beta_1 z$.

$\phi(z)$ the Soliton's phase, depends only on $z$.

$\phi'(\tau)$ is unchanged with time,

and

$a$ is the maximal amplitude. That is,

$$0 < u(\tau) \le 1.$$

Substituting in the Nonlinear Schrodinger equation,

$$au(\tau)e^{i\phi(z)}i\phi'(z) - i\frac{1}{2}\beta_2 au''(\tau)e^{i\phi(z)} + i\frac{1}{2}a^3\omega_0\varepsilon_0 n_0 n_2 u^2 u e^{i\phi(z)} = 0.$$

$$u(\tau)\phi'(z) - \frac{1}{2}\beta_2 u''(\tau) + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^3 = 0$$

$$\phi'(z) = \frac{1}{2}\beta_2 \frac{u''(\tau)}{u(\tau)} - \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^2 .$$

Thus, both sides equal to a constant $\alpha$. Hence,

$$\phi(z) = \alpha z ,$$

and

$$\alpha = \frac{1}{2}\beta_2 \frac{u''(\tau)}{u(\tau)} - \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^2$$

Therefore,

$$u''(\tau) = \frac{1}{\beta_2}\left(2\alpha u + a^2\omega_0\varepsilon_0 n_0 n_2 u^3\right)$$

is the Soliton's Envelope Equation.

# 6.

# The Fundamental Soliton solution

To solve the Soliton envelope equation

$$u''(\tau) = \frac{1}{\beta_2}\left(2\alpha u + a^2\omega_0\varepsilon_0 n_0 n_2 u^3\right),$$

multiply it by the integration factor $2u'(\tau)$.

$$2u'u'' = \frac{1}{\beta_2}\left(4\alpha uu' + 2a^2\omega_0\varepsilon_0 n_0 n_2 u^3 u'\right)$$

$$D_\tau\left(u'\right)^2 = \frac{1}{\beta_2}\left(2\alpha D_\tau\left(u^2\right) + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 D_\tau\left(u^4\right)\right)$$

Integrating both sides with respect to $\tau$,

$$\left(u'\right)^2 = \frac{1}{\beta_2}\left(2\alpha u^2 + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^4 + C_1\right).$$

Assuming that if $\tau \to \infty$, then $u' \to 0$, and $u \to 0$, we have

$$C_1 = 0,$$

and

$$\left(u'\right)^2 = \frac{1}{\beta_2}\left(2\alpha u^2 + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^4\right)$$

$$= \frac{1}{\beta_2}u^2\left(2\alpha + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^2\right)$$

At $\tau = 0$, $u(\tau)$ peaks to $u = 1$. Then, $u' = 0$, and the equation becomes

$$0 = \frac{1}{\beta_2}\left(2\alpha + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2\right)$$

Hence,

$$2\alpha = -\frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 .$$

Substituting into the Soliton envelope equation

$$(u')^2 = \frac{1}{\beta_2}u^2\left(-\frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 + \frac{1}{2}a^2\omega_0\varepsilon_0 n_0 n_2 u^2\right)$$

$$= \frac{1}{-2\beta_2}a^2\omega_0\varepsilon_0 n_0 n_2 u^2\left(1 - u^2\right)$$

In optical Fibers $\beta_2 < 0$, and since the maximum of $u$ is $1$, the square root of the right hand side is real positive, and we have

$$u' = \underbrace{a\sqrt{\frac{1}{-2\beta_2}\omega_0\varepsilon_0 n_0 n_2}}_{\frac{1}{T}}\, u\sqrt{1-u^2} .$$

Denoting

$$a\sqrt{\frac{1}{-2\beta_2}\omega_0\varepsilon_0 n_0 n_2} \equiv \frac{1}{T},$$

$$u' = \frac{1}{T}u\sqrt{1-u^2} ,$$

where $T > 0$.

$$\frac{du}{u\sqrt{1-u^2}} = \frac{1}{T}d\tau .$$

Integrating both sides,

$$\int \frac{du}{u\sqrt{1-u^2}} = \frac{1}{T}\int d\tau + C_2.$$

From Integration Tables, [Spiegel, p.69],

$$-\log \frac{1+\sqrt{1-u^2}}{u} = \frac{\tau}{T} + C_2.$$

At $\tau = 0$, $u(\tau)$ peaks to $u = 1$. Then,

$$-\log \frac{1+\sqrt{1-1^2}}{1} = \frac{0}{T} + C_2$$

That is,

$$0 = C_2,$$

and

$$\frac{1+\sqrt{1-u^2}}{u} = e^{-\frac{\tau}{T}},$$

$$1-u^2 = (ue^{-\frac{\tau}{T}} - 1)^2,$$

$$= u^2 e^{-2\frac{\tau}{T}} - 2ue^{-\frac{\tau}{T}} + 1$$

$$0 = u\left[u(e^{-2\frac{\tau}{T}} + 1) - 2e^{-\frac{\tau}{T}}\right]$$

$$u(e^{-2\frac{\tau}{T}} + 1) = 2e^{-\frac{\tau}{T}}$$

$$u(e^{-\frac{\tau}{T}} + e^{\frac{\tau}{T}}) = 2$$

$$u = \frac{2}{e^{-\frac{\tau}{T}} + e^{\frac{\tau}{T}}}$$

$$= \frac{1}{\cosh\left(\dfrac{\tau}{T}\right)}$$

$$= \frac{1}{\cosh\left(\dfrac{t - z / v_g}{T}\right)}$$

Since $a\sqrt{\frac{1}{-2\beta_2}\omega_0\varepsilon_0 n_0 n_2} = \frac{1}{T}$,

$$a = \frac{1}{T}\sqrt{\frac{-2\beta_2}{\omega_0\varepsilon_0 n_0 n_2}} \,,$$

and

$$\alpha z = -\frac{1}{4}a^2\omega_0\varepsilon_0 n_0 n_2 z$$

$$= \frac{\beta_2}{2T^2}z\,.$$

Thus, the Fundamental Soliton solution to the Non-linear Schrödinger equation is

$$f(z,\tau,T) = \frac{1}{T}\sqrt{\frac{-2\beta_2}{\omega_0\varepsilon_0 n_0 n_2}}\frac{1}{\cosh\left(\dfrac{t - z / v_g}{T}\right)}e^{-i\frac{\beta_2}{2T^2}z}\,.$$

*The Soliton's amplitude depends on its width* $T$ *.*

# *Delta Function*

We aim to show that the Fundamental optical Soliton can be modeled by a propagating Delta Function.

The description of the Delta Function as the limit of a Delta sequence leads to divergent integrals, and to meaningless equality to the inconceivable infinity.

In the Calculus of Limits, the Delta function is not defined, and its power to describe physical singularities is restricted.

Using infinitesimals, and infinite hyper-real numbers, the Delta Function can be defined on the hyper-real line, an infinite dimensional line that has room for infinitesimals, and their reciprocals, the infinite hyper-reals.

Then, the Delta function is the whole infinite Delta sequence, and serves to model singularities.

In the next sections, we sum up the main points necessary for the definition, and the application of the Delta function

# 7.

# Hyper-real Line

Each real number $\alpha$ can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.
15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or

to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

# 8.

# Hyper-Real Integral

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the $x$'s that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal $dx$, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

## 8.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = .... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

**8.2** $$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

**8.3** $$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

**8.4** *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

# 9.

# Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real $0$ is the sequence $\left\langle 0,0,0,... \right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of $dx$.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle,...$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle,...$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal

$dx$, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than $\infty$

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

   where $\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from $0$ to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to $0$.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

**9.** $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

# 10.

# The Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ = *the inverse Fourier Transform of the unit function* $1$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x} d\nu\,, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite hyper-real

$$\left.\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

## 5. Fourier Integral Theorem in Infinitesimal Calculus

*If* $f(x)$ *is hyper-real function,*

*Then,*

- *the Fourier Integral Theorem holds.*
- $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ *converges to* $F(\alpha)$
- $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ *converges to* $f(x)$

# 11.

# The Laplace Transform

In [Dan7], we have shown that

1. The Delta function $\delta(t)$, for $t \geq 0$ is represented by Sequence

   $\delta_n(t) = ne^{-nt}\chi_{[0,\infty)}(t)$.

2. If $i_n = \dfrac{1}{n}$, $\quad \delta(x) = \left\langle e^{-t}\chi_{[0,\infty)}(t), 2e^{-2t}\chi_{[0,\infty)}(t), 3e^{-3t}\chi_{[0,\infty)}(t), ... \right\rangle$

3. $\mathcal{L}\left\{\delta(t)\right\} = 1$

4. $\delta(t)$ $=$ *the inverse Laplace Transform of the unit function* $1$

   $$= \frac{1}{2\pi i}\int_{s=-i\infty}^{s=i\infty} e^{st}ds\,.$$

5. $$\frac{1}{2\pi i}\int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st}ds\Bigg|_{t=0} = \frac{1}{dt} = \text{an infinite hyper-real}$$

   $$\int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st}ds\Bigg|_{t\neq 0} = 0\,.$$

6. Laplace Integral Theorem

   *If* $f(t)$ *is hyper-real function,*

   *Then,*

   *the Laplace Integral Theorem holds.*

$$f(t) = \frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st} \left( \int_{\tau=0}^{\tau=\infty} e^{-s\tau} f(\tau) d\tau \right) ds$$

- $\int_{t=0}^{t=\infty} e^{-st} f(t) dt$, ***converges to*** $F(s)$

- $\frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st} F(s) ds$ ***converges to*** $f(t)$.

# 12.

# Pulse Compression and Delta

Pulse Compression is used to generate Optical Solitons, with narrower width $T$, and larger amplitudes proportional to $\frac{1}{T}$ that are used to generate even more compressed Solitons.

An infinite sequence of compressed pulses with indefinitely narrowing widths and growing amplitudes constitutes a Delta function.

The narrowing widths include less carrier cycles, and require higher carrier frequencies. At X-ray frequencies the laser mirrors have to be replaced by dielectric mirrors, and as the frequency increases, at Attosecond (Atto=$10^{-18}$ ) pulses, no mirrors will do, and a different compression method has to be applied.

Physically, such Delta Pulse Sequence is unattainable.

However, similarly to considering the sizes of a charge or a point light source as infinitesimal and representing them by a Delta function, we shall ignore the physical limitations on realizing an ideal infinitesimal-width Optical Soliton, and model the sequence of narrowing Solitons by a Delta function.

### 12.1 Pulse Width, and Delta

A system that combines dispersion with self-phase modulation will generate a narrower output pulse.

In [Siegman, p.393]

an input pulse of 5,900 $f\sec$, (femto=$10^{-15}$)

enters a 3 $meter$ optical fiber, and is broadened to

a pulse of 10,000 $f\sec$.

That pulse enters a diffraction grating that compresses it into

an output pulse of 200 $f\sec$.

That pulse enters a 0.55 $meter$ optical fiber, followed by a diffraction grating, that compresses it into

an output pulse of 90 $f\sec$.

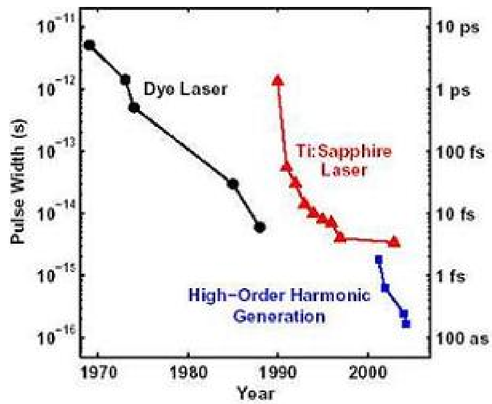
Pulses of 90 $f\sec$ generated in a Dye laser have been compressed to 30 $f\sec$-the shortest width attained in 1986.

A Titanium-Sapphire laser [W-1] generates pulses in the range of 100 $f\sec$, down to 10 $f\sec$. The shortest width attained in 1999 was 5.5 $f\sec$.

In 2008, 2.5 $f\sec$ pulses, containing only one or two cycles of the carrier, were used in the [W-3] experiment.

Narrower pulses were generated, but not in optical fibers, and not as Solitons with amplitude that grows as the width narrows.

[W-2] describes the evolution up to 2008, of Ultrashort Pulses into the Attosecond range (Atto=$10^{-18}$ ).




**Evolution of ultrashort light pulses up to 2008. (from [W-2])**

The Attosecond flushes were attained, by firing ultrashort laser pulses into a cloud of neon gas, generating carrier in the extreme ultraviolet range.

In [W-2] such Attosecond flushes from Argon, were used to stroboscope-light and film the motion of an electron, after it was separated from an atom, in time period of a single carrier cycle.

By [W-3], the shortest such flush in 2011 is 80 Attoseconds.

In Comparison, an electron circles the nucleus in 150 Attosecond.

## 12.2 Pulse Power, and Delta

By [W-6], A carrier wave from an Argon or Nd:YVO4 laser with average output power of 0.5 to 1.5 Watts, pumps a Titanium-sapphire laser of mode-locked oscillator type.

The Ti:Sapphire oscillator generates pulses with duration as short as 5 $f\sec$.

According to [Weiner,p.405], at 100 $f\sec$,

the peak power of the pulse is $10^4$ Watts.

This Corresponds to maximum intensity $10^8$ W/m$^2$ =$10^{12}$ W/cm$^2$.

By [W-6], this pulse is supplied to a Titanium-sapphire laser of Chirped-pulse amplifier type, designed to withstand the damaging power of the pulse.

The Ti:Sapphire laser amplifier may use Chirped mirrors [W-7] to guide the beam several times through the crystal, while keeping the pulse width below the damage threshold, or it may use optical switches, to insert the pulse in the cavity, and retrieve it later.

The Ti:Sapphire amplifier generates pulses with duration of 20 to 100 $f\sec$ with energy of 5 $mJouls$.

At 100 $f\sec$, the peak power of the pulse is 50 $GegaWatts$. An electric power plant peaks to 1 GW.

This corresponds to maximum intensity

$$(25)10^{20}\ \mathrm{W/m^2} = (25)10^{24}\ \mathrm{W/cm^2}$$

Thus, the Delta function is suitable to describe short energetic Optical Fiber Solitons.

# 13.

# Solitons and Delta Sequence

## 13.1 Soliton's Delta

The Soliton

$$f(z,t,T) = \frac{1}{T}\sqrt{\frac{-2\beta_2}{\omega_0\varepsilon_0 n_0 n_2}}\frac{1}{\cosh\left(\frac{t - z / v_g}{T}\right)}e^{-i\frac{\beta_2}{2T^2}z},$$

has the envelope

$$\frac{1}{T}\sqrt{\frac{-2\beta_2}{\omega_0\varepsilon_0 n_0 n_2}}\frac{1}{\cosh\left(\frac{t - z / v_g}{T}\right)},$$

where

$$n_2 = \frac{3}{4}\frac{\chi^{(3)}}{c_0\varepsilon_0 n_0^2},$$

$n(\omega, E_0) \approx n_0 + n_2\left(\frac{1}{2}c_0\varepsilon_0 n_0 E_0^2\right)$ is the refractive index,

$$\beta \equiv \frac{\omega_0}{c_0}n(\omega, E_0),$$

$$\beta_2 \equiv \left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\substack{\omega=\omega_0\\ E_0=0}},$$

$v_g$ is the envelope velocity,

$T$ is the Soliton's width.

Denote

$$\gamma \equiv \sqrt{\frac{-2\beta_2}{\omega_0 \varepsilon_0 n_0 n_2}},$$

$$\tau \equiv t - z / v_g,$$

$$T = dz.$$

The Delta function representing this envelope is the Hyper-real function

$$\frac{1}{dz}\gamma \frac{1}{\cosh\left(\dfrac{t - z / v_g}{dz}\right)}.$$

For $dz = \left\langle \frac{1}{n} \right\rangle$, this Delta function is represented by the sequence

$$\left\langle \frac{\gamma}{\cosh\left(t - z / v_g\right)}, \frac{2\gamma}{\cosh 2\left(t - z / v_g\right)}, \frac{3\gamma}{\cosh 3\left(t - z / v_g\right)}, \ldots \right\rangle$$

**13.2** Each $\delta_n(\tau) = \dfrac{n}{\pi}\dfrac{1}{\cosh n\tau}$

- has the sifting property $\displaystyle\int_{\tau=-\infty}^{\tau=\infty} \delta_n(\tau) d\tau = 1$
- is continuous
- peaks at $\tau = 0$ to $\delta_n(0) = \dfrac{n}{\pi}$

*Proof*:

By [Spiegel, p.88],

$$\int_{\tau=-\infty}^{\tau=\infty} \frac{n}{\pi}\frac{1}{\cosh n\tau}d\tau = \frac{n}{\pi}\frac{2}{n}\arctan\left(e^{n\tau}\right)\Big|_{\tau=-\infty}^{\tau=\infty}$$

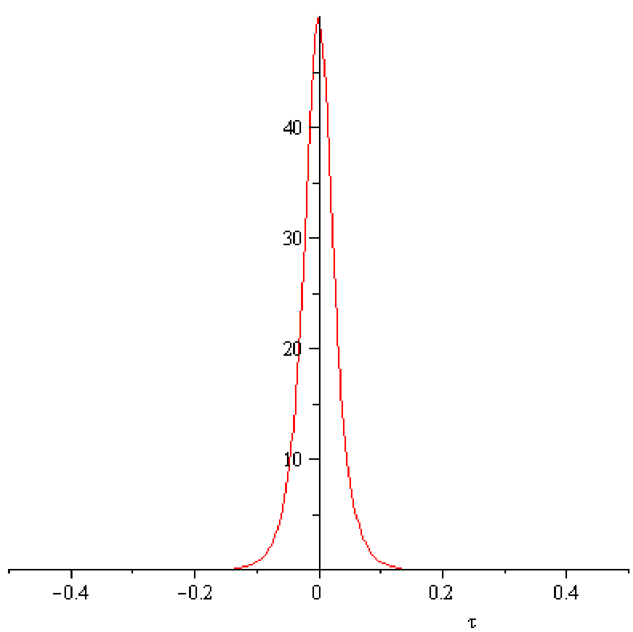
$$= \frac{2}{\pi}[\underbrace{\arctan(\infty)}_{\pi/2} - \underbrace{\arctan(0)}_{0}]$$

$$= 1.\square$$

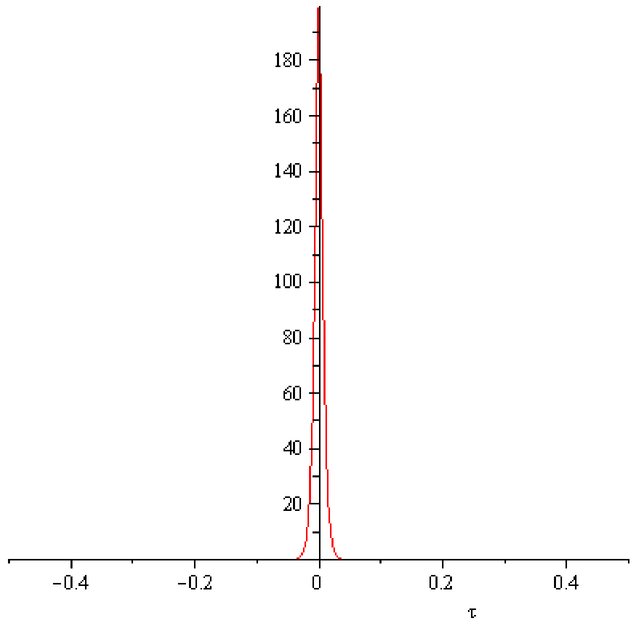
Thus, the sequence represents the hyper-real Delta Function

**13.3** $$\delta(\tau) = \left\langle \frac{1}{\cosh\tau}, \frac{2}{\cosh 2\tau}, \frac{3}{\cosh 3\tau}, \ldots \right\rangle$$

$plot\left(\frac{50}{\cosh(50\,\tau)}, \tau = -0.5\,..0.5\right)$ plots in Maple, the 50th component,



$plot\left(\dfrac{200}{\cosh(200\,\tau)}, \tau = -0.5\,..0.5\right)$ plots in Maple the $200^{th}$ component,



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